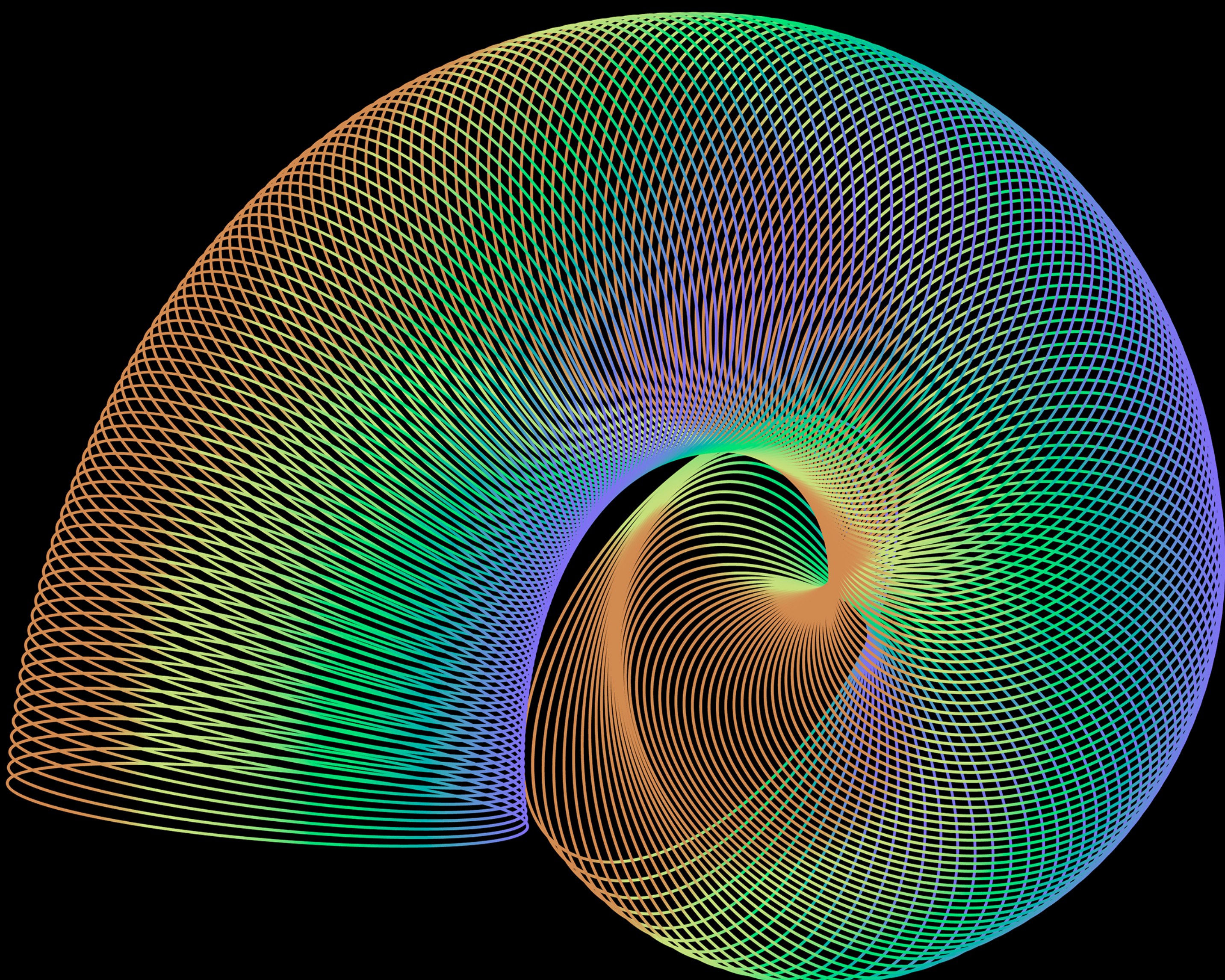
Edvantage Math

# AP® Calculus AB



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# AP Calculus AB

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AP Calculus AB


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# Contents

## AP Calculus AB

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# Unit 1: Differential Calculus

This unit focuses on the following AP Big Idea from the College Board:

**Big Idea 1: The idea of limits is essential for discovering and developing important ideas, definitions, formulas, and theorems in calculus**

**By the end of this unit, you should be able to:**

- Express limits symbolically using correct notation
- Interpret limits expresses symbolically
- Estimate limits of functions
- Determine limits of functions
- Deduce and interpret behavior of functions using limits
- Analyze functions for intervals of continuity or points of discontinuity
- Determine the applicability of important calculus theorems using continuity

By the end of this unit, you should know the meaning of these **key terms:**

- Continuity at a point
- Continuous function
- Difference rule
- End behavior models (left-end and right-end)
- Extended function
- Horizontal asymptote
- Infinite discontinuity
- Intermediate Value Theorem (for continuous functions)
- Jump discontinuity
- Left-hand limit
- Limit
- One-sided limit
- Oscillating discontinuity
- Product rule
- Quotient rule
- Right-hand limit
- Sandwich Theorem
- Sum rule
- Two-sided limit
- Vertical asymptote

*When the Mars Rover lands, it's important to know its velocity at all times.*

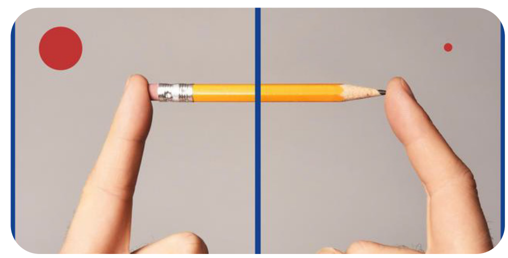
# Chapter 1: Analyzing Functions Using Limits

***Limits tell you where you're going, even if you never actually get there***

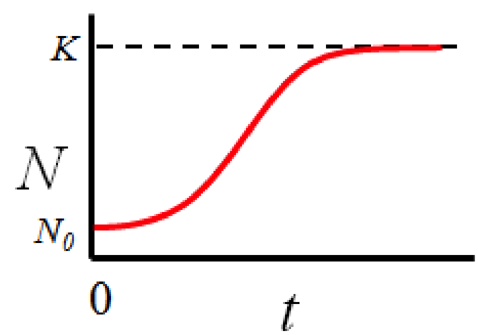
Students often wonder "What is Calculus?" Calculus uses the logic of mathematics to analyze situations involving continuous change by breaking them down into the smallest components (differential calculus) and then rebuilding them (integral calculus). As such, Calculus is considered the science and the art of change. It provides a mechanism to understand the world around us, make predictions, and interpret behavior. One type of behavior involves a limit. What is a limit and how is it useful?

The mathematical answer is that a limit is the y-value that a function, f(x), approaches as the value of x approaches some number. A few "real life" examples can be found in:

**Physics** - pressure at a point is calculated as the average pressure (force per unit area) applied to an area that is shrinking to zero (i.e., shrinking to a point).



**Ecology** - sustainable population (or carrying capacity) is often extrapolated by determining, over an extended period of time (i.e., infinite), the upper bound on some population that can be sustained by a given ecosystem.




**Chemistry** - if you drop an ice cube into a glass of warm water and measure the temperature against time, the temperature will eventually approach the room temperature where the glass is stored. Determining the final temperature is a limit as time approaches infinity.

**A limit describes the behavior of a function at a particular point based on the points around it.**

## EXPLORING THE BIG IDEA

In this chapter you should be able to:

- Determine the limits of functions using limit theorems.
- Interpret limits expressed in analytic notation.
- Represent limits analytically using correct notation.

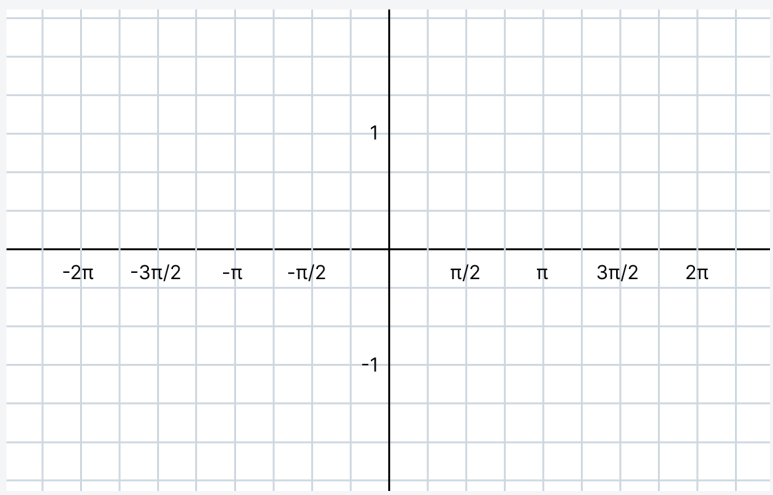
# 1.1 Rates of Change and Limits – Determining Limits Graphically

## Warm Up

The concept of a limit is CENTRAL to calculus. Limits can be used to describe how a function behaves as the independent variable moves towards a certain value.

Consider the functions $f(x)=\sin x$ and $g(x)=x$

Graph $f(x)=\sin x$ on $[-2\pi,2\pi]$, using an appropriate range. On the same grid graph $g(x)=x$.
(Be careful…when does $x=1$ ?)




When does $g(x)=f(x)$?

What are the values of $g(x)$ and $f(x)$ at the point of intersection?

How do $g(x)$ and $f(x)$ behave near $x=0$?

What do you think the value of the quotient function $y=\dfrac{\sin x}{x}$ is at the point of intersection?

Look graphically and numerically:
Using your calculator, graph both f(x) and g(x), then complete the tables of values below.

| *X* | *g*(*x*) | *f*(*x*) |
|---|---|---|
| -0.3 | | |
| -0.2 | | |
| -0.1 | | |
| 0 | | |
| 0.1 | | |
| 0.2 | | |
| 0.3 | | |

| *X* | *g*(*x*) | *f*(*x*) |
|---|---|---|
| -0.03 | | |
| -0.02 | | |
| -0.01 | | |
| 0 | | |
| 0.01 | | |
| 0.02 | | |
| 0.03 | | |

Consider the y-value as the x-value gets closer and closer to 0.

How do the functions behave as x approaches 0? (use the ZOOM function on your calculator to explore)

What value will the quotient approach as the values of the functions become more and more alike?

**Definition: Indeterminate form**

If functions $g(x)$ and $f(x)$ are both 0 at $x = a$ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ cannot be found by substituting $x = a$ as the result would be $\frac{0}{0}$. Substitution produces a meaningless expression known as an indeterminate form.

Other indeterminate forms include: $\frac{\infty}{\infty}, (\infty)(0), \infty - \infty, 1^{\infty}, 0^0, \infty^0$ .

To determine the value of an indeterminate form, other methods, such as graphing, analyzing numerically or using algebraic manipulations, must be used.

### Example 1: Determining a Limit Graphically and Numerically

Find the value that $f(x)=\frac{\sin x}{x}$ approaches as x approaches 0 $\left(x\to 0\right)$. Use a graph and a table of values to justify your answer.

| How to Do it... | What to Think About |
|---|---|
| | How does the function behave as *x* gets closer to zero from the left? From the right?<br><br>What is the value of the function at $x = 0$ ?<br><br>What value does the function approach as *x* gets closer to zero from the left? From the right? |

**Definition: Limit**

$\lim_{x\to a} f(x) = L$ is read, "the limit of *f* of *x* as *x* approaches *a* equals *L*". It means that as *x* gets closer and closer to *a* (from the left-side and the right-side of *a*), the function's *y* – value gets closer and closer to the number *L*.

**Your Turn ...**

Compare $y=\sin 2x$ and $y=2x$ graphically then graph $f(x)=\dfrac{\sin 2x}{2x}$ and find the value that y approaches as $x \to 0$ . Confirm using a table of values.

| $x$ | $y$ |
|---|---|
| -0.03 | |
| -0.02 | |
| -0.01 | |
| 0 | |
| -0.01 | |
| -0.02 | |
| -0.03 | |

**Example 2: The Existence of a Limit Versus the Existence of the Function at a Point.**

Given $f(x)=\begin{cases}3, x=2\\ 2x+1, x\neq 2\end{cases}$ determine $\lim\limits_{x\to 2} f(x)$. Does $\lim\limits_{x\to 2} f(x)$ equal to $f(2)$ ?

**How to do it...**

**What to Think About**

How does the function behave as x gets closer to 2 from the left? From the right?

What is the value of the function at x = 2?

Is the function continuous at x = 2?

## Your Turn ...

For each of the following functions, determine if $\lim_{x \to a} f(x) = f(a)$

a) $f(x) = \begin{cases} x^2 - 1, x \le 1 \\ \sqrt{x} - 1, x < 1 \end{cases}$ at $a = 1$

b) $f(x) = \begin{cases} x^2 - 1, x < 1 \\ 1, x = 1 \\ \sqrt{x} - 1, x > 1 \end{cases}$ at $a = 1$

**Did You Know?**
The modern notation of placing the arrow below the limit symbol is attributed to the English mathematician G.H. Hardy, who used it in his 1908 book A Course of PureMathematics.

**SUMMARY**
The value of a limit is the *y* – value that the function approaches as $x \to a$. The existence of a limit as $x \to a$ never depends on whether the function is defined or undefined at $x = a$. The limit exists when the function approaches the same *y* – value on both sides of $x = a$.

## 1.1 Practice

For questions 1 to 8, determine the limit graphically.

1. $\lim\limits_{x \to 1} \dfrac{x+1}{x^2-1}$

2. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{3-x}-\dfrac{1}{3}}{x}$

3. $\lim\limits_{x \to 3} \dfrac{x^2-5x+6}{x^2-9}$

4. $\lim\limits_{x \to 3} \dfrac{(6-x)^2-9}{x-3}$

5. $\lim\limits_{x \to 0} \dfrac{7x^3-3x^2}{5x^4-9x^2}$

6. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

7. $\lim_{x\to 0}\frac{x+\sin x}{x}$

8. $\lim_{x\to 0}\frac{\sin^2 x}{x}$

For questions 9 to 12, use a graph to show that the limit does or does not exist

9. $\lim_{x\to 1}\frac{x^2-4}{x-1}$

10. $\lim_{x\to 2}\frac{x^2+1}{x^2-4}$

11. $\lim_{x\to 3}\frac{1}{(x-3)^2}$

12. $\lim_{x\to 8}\frac{x^2+16x+64}{x+8}$

# 1.2 Exploring Limits

## Warm Up

Consider the following functions:

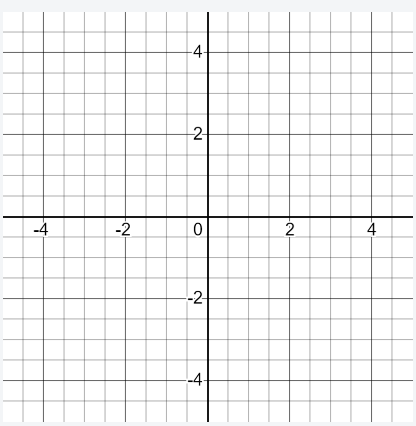
a) $h(x) = x + 1$

b) $f(x) = \dfrac{x^2 - 1}{x - 1}$

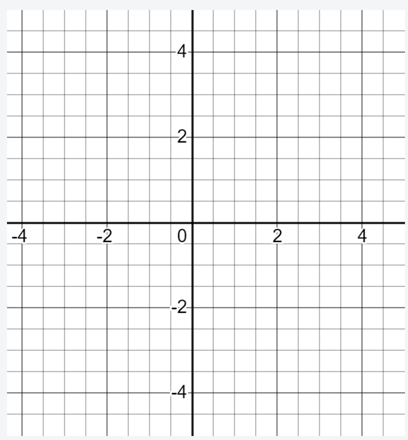
c) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, x \neq 1 \\ 1, x = 1 \end{cases}$

Without a graphing calculator sketch the graph of each function. Make sure to consider the points where the function does not exist.

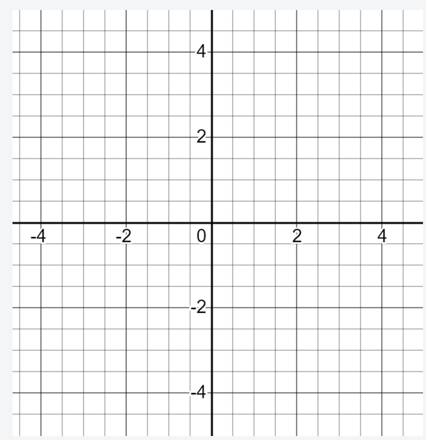
a)




b)




c)




Using the graphs, determine each of the following limits and confirm algebraically.

a) $\lim_{x \to 1} h(x) =$

b) $\lim_{x \to 1} f(x) =$

c) $\lim_{x \to 1} g(x) =$

Why does factoring help to determine the limit algebraically?

**Did You Know?**

A **removable discontinuity** (or point discontinuity) occurs when a function has a hole at one point $c$ on an open interval such that $\lim_{x \to c} f(x) = L$, but $f(c) \neq L$ or $f(c)$ does not exist. You will explore the concept of continuity in Section 1.5.

## Example 1: Determining the Limit of Function

Find the limit of each of the following functions, if it exists.

a) $\lim_{x \to 2} x^3 + 4x^2 - 3$  b) $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$  c) $\lim_{x \to 1} \frac{1}{x - 1}$  d) $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25}$

| How to Do it... | What to Think About |
| --- | --- |
| | How does the function behave as $x$ gets closer to zero from the left? From the right?<br><br>What is the value of the function at $x = a$ ?<br><br>What value does the function approach as $x$ gets closer to $a$ from the left? From the right? |

## Your Turn...

Find the limit of each of the following functions, if it exists.

a) $\lim_{x \to 1} \frac{x^4 + x^2 - 1}{x^2 + 5}$

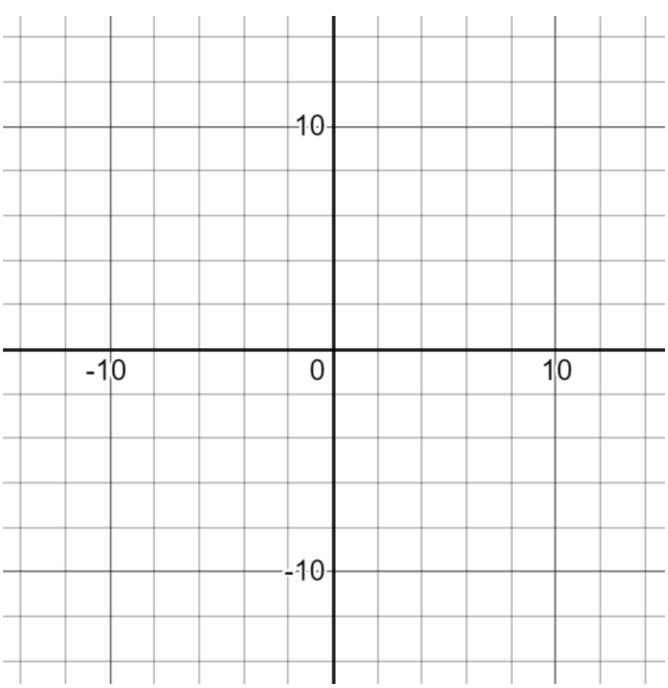
b) $\lim_{x \to -2} \frac{3}{x+2}$

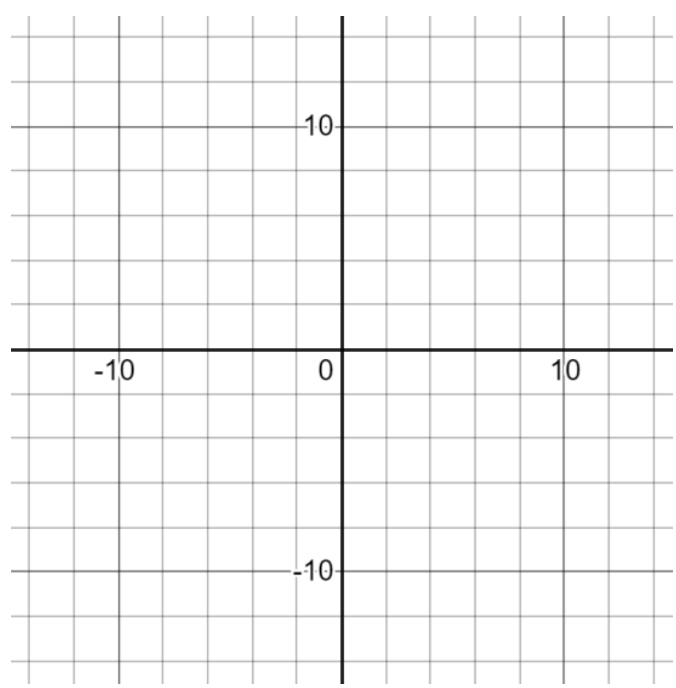
c) $\lim_{x \to 3} \frac{x-3}{x^2 - 5x + 6}$

d) $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$

e) $\lim_{h \to 0} \frac{\frac{1}{h+1} - 1}{h}$

Look at b) and c) graphically. What do you think makes a limit exist?



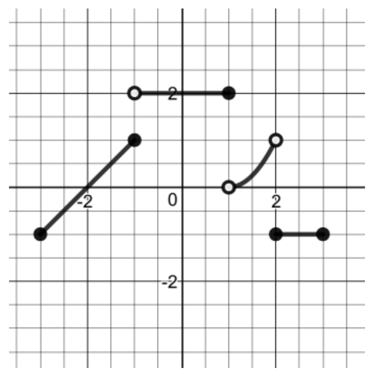



**Definition: One-sided limit**

The existence of a limit depends on the one-sided limits about $x = a$. So when the $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$, then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = L$ where $L \in \mathbb{R}$. When the right-hand limit equals the left-hand limit at a point, the limit exists.

## Example 2: One-sided and Two-sided Limits

Given the graph of *f* determine each limit




### How to Do It...

### What to Think About

What does the notation "$x \to 1^+$" mean?

What does the notation "$x \to 1^-$" mean?

Is the left-side limit the same as the right-side limit?

What does the abbreviation "DNE" represent?

## Your Turn...

1) Given the graph of $g$ determine each limit.

a) $\lim_{x\to 1^+} g(x) =$

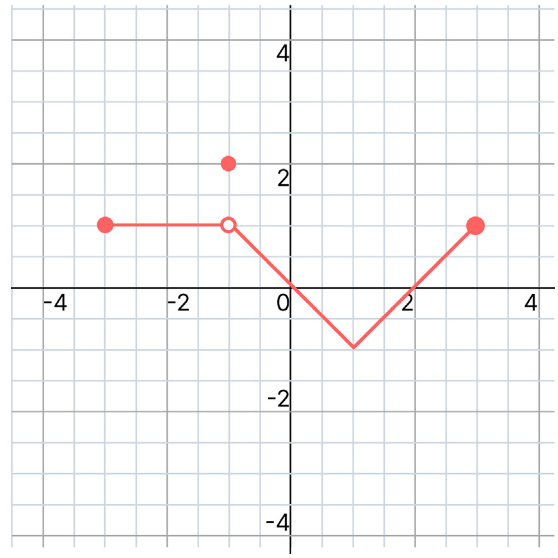
b) $\lim_{x\to 1^-} g(x) =$

c) $\lim_{x\to 1} g(x) =$

d) $\lim_{x\to 0} g(x) =$

e) $\lim_{x\to -1^+} g(x) =$

f) $\lim_{x\to -1^-} g(x) =$

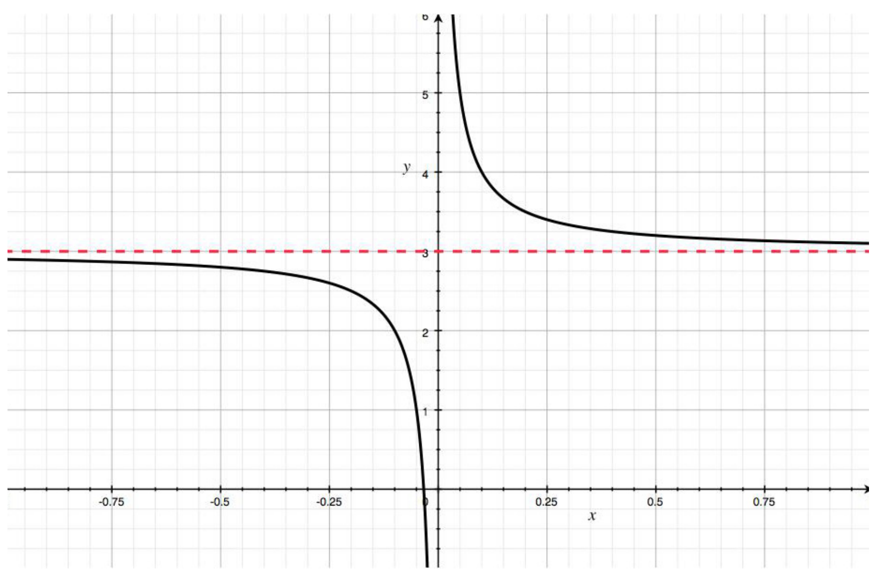



2) Given the graph of $f$ determine each limit.

a) $\lim_{x\to 0^+} f(x) =$

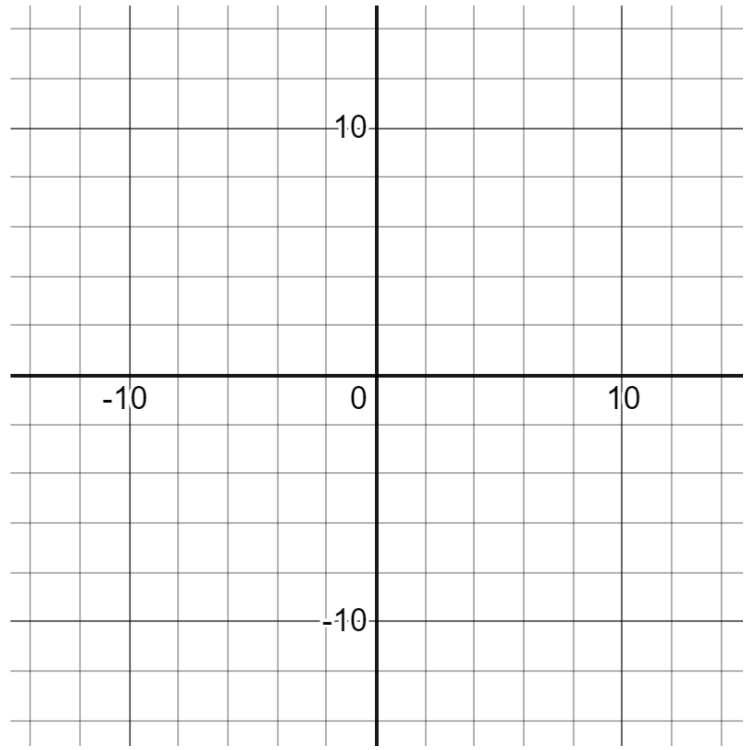
b) $\lim_{x\to 0^-} f(x) =$

c) $\lim_{x\to 0} f(x) =$



3) Given the following function, sketch the graph of the function and then use one-sided limits to show that the $\lim_{x \to 2} f(x)$ does not exist.

$$f(x) = \begin{cases} 3 - x, x < 2 \\ 5, x = 2 \\ x, x \geq 2 \end{cases}$$



## Example 3: The Greatest Integer Function

$y = [x]$ or $y = \text{int}(x)$, is defined to be the largest integer to the "left" of $x$ on the number line (or the greatest integral part of $x$). What is the result when the greatest integer function is applied to each of the following?

a) int(2.3) b) int(2.9) c) int(3) d) int(–1.2) e) int(–3.5) f) int(0.2)

### How to Do it...

### What to Think About

What is the greatest integer less than the number 2.3?

What is the greatest integer less than –1.2? Try plotting this on a number line.

### Did You Know?

The greatest integer function is commonly used when calculating utility bills (gas, electricity, water, etc.) as well as postage rates.

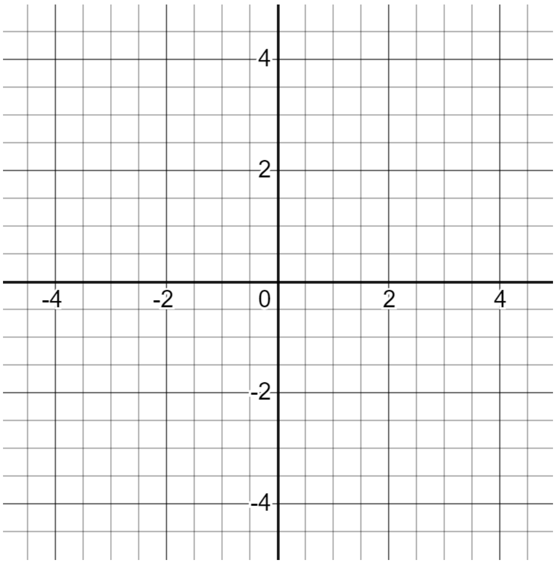
**2015 Canada Postage for Lettermail, Letter Post, Light Packer (LP), Small Packet Air (SP)**
Maximum dimensions: 380 mm x 270 mm x 20 mm for lettermail, letter post, light packet

| Weight | Canada | USA | International |
|---|---|---|---|
| ≤30g | 0.85+tax | 1.20+tax | 2.50+tax |
| ≤50g | 1.20+tax | 1.80+tax | 3.60+tax |
| ≤100g | 1.80+tax | 2.95+tax | 5.95 |
| ≤150g | - | - | - |
| ≤200g | 2.95+tax | 5.00(LP)<br>5.15 | 7.50(LP)<br>10.30 |
| ≤250g | - | 8.13(SP) | 9.49-10.67(SP) |
| ≤300g | 4.10+tax | 7.0(LP) | 11.50(LP) |
| ≤400g | 4.70+tax | - | - |
| ≤500g | 5.05+tax | 10.30<br>10.79(SP)<br>11.75(LP) | 18.98-21.68(SP)<br>20.60<br>21.00(LP) |
| ≤1000g | - | 16.23(SP) | 37.22-40.44(SP) |
| ≤1500g | - | - | 46.57-52.78(SP) |
| ≤2000g | - | - | 55.93-61.40(SP) |

## Your Turn...

1. Graph the greatest integer function (also called a step function):




2. Use the graph to help determine the following limits:

a) $\lim_{x \to 0.6} \text{int}(x) =$

b) $\lim_{x \to -1.4} \text{int}(x) =$

c) $\lim_{x \to 1.5} \text{int}(x) =$

d) $\lim_{x \to 1^+} \text{int}(x) =$

e) $\lim_{x \to 1^-} \text{int}(x) =$

f) $\lim_{x \to 1} \text{int}(x) =$

**SUMMARY: For the greatest integer function,** ( $n \in Z$ , $n$ is an integer)

$\lim_{x \to n^+} [x] = n$

$\lim_{x \to n^-} [x] = n - 1$

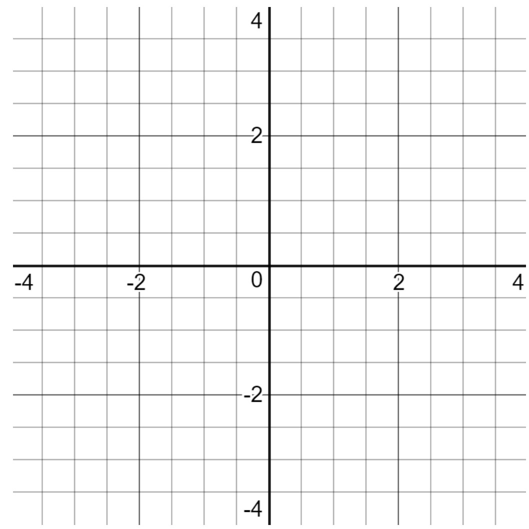
$\lim_{x \to n} [x] = DNE$

## 1.2 Practice

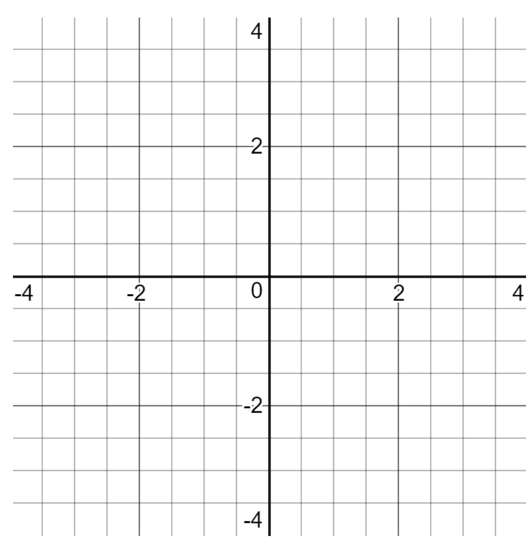
1. What is $\lim_{x \to c}(2x^3 - 3x^2 + x - 1)$?

Determine each limit by substitution and support the result graphically.

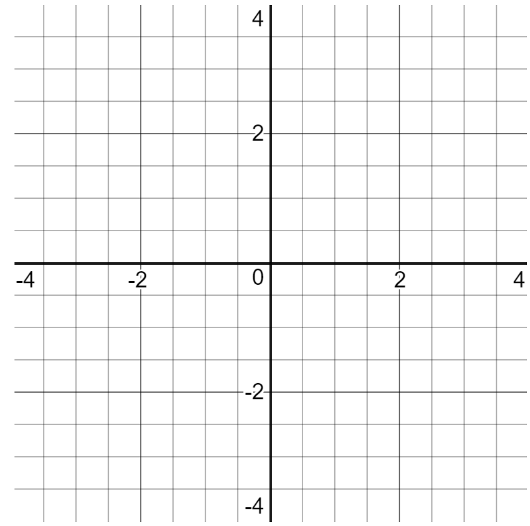
2. $\lim_{x \to -\frac{1}{2}} 4x^2(2x-1)$



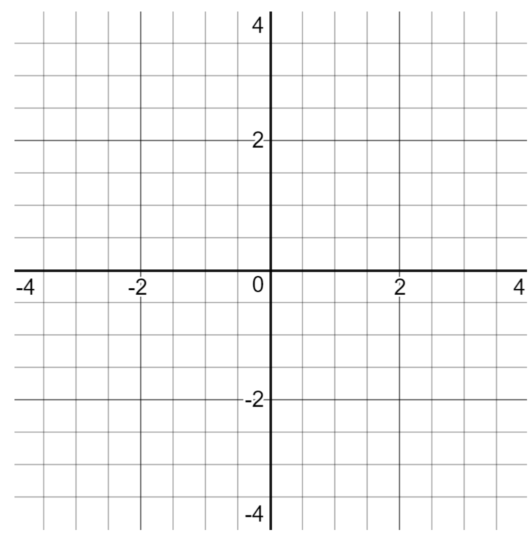

3. $\lim_{x \to 5}(x-6)^{2020}$



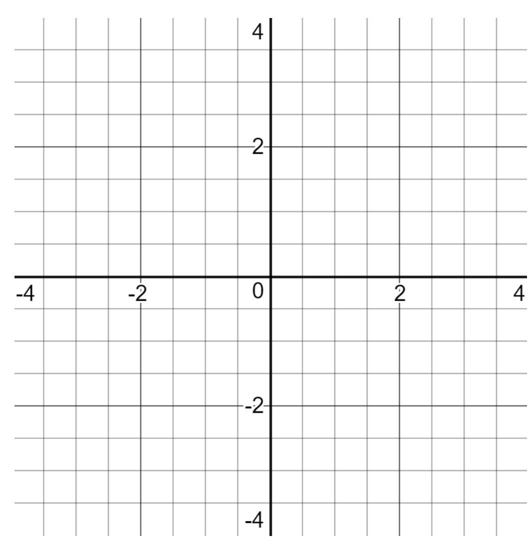
4. $\lim_{x \to 1}(x^4 - 3x^3 + 2x - 5)$




5. $\lim_{y \to -1} \frac{y^2 + 5y + 6}{y^2 + 1}$




6. $\lim_{x \to \frac{1}{3}} \text{int}(x)$



7. Explain why you cannot use substitution to determine each limit. Use a graph to support your explanation

a. $\lim_{x \to -2} \sqrt{x-3}$

**Did You Know?**

The signum function is a mathematical function that extracts the sign of a real number. It is often represented as

$$\frac{|x|}{x} = \text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

b. $\lim_{x \to 0} \frac{|x|}{x}$

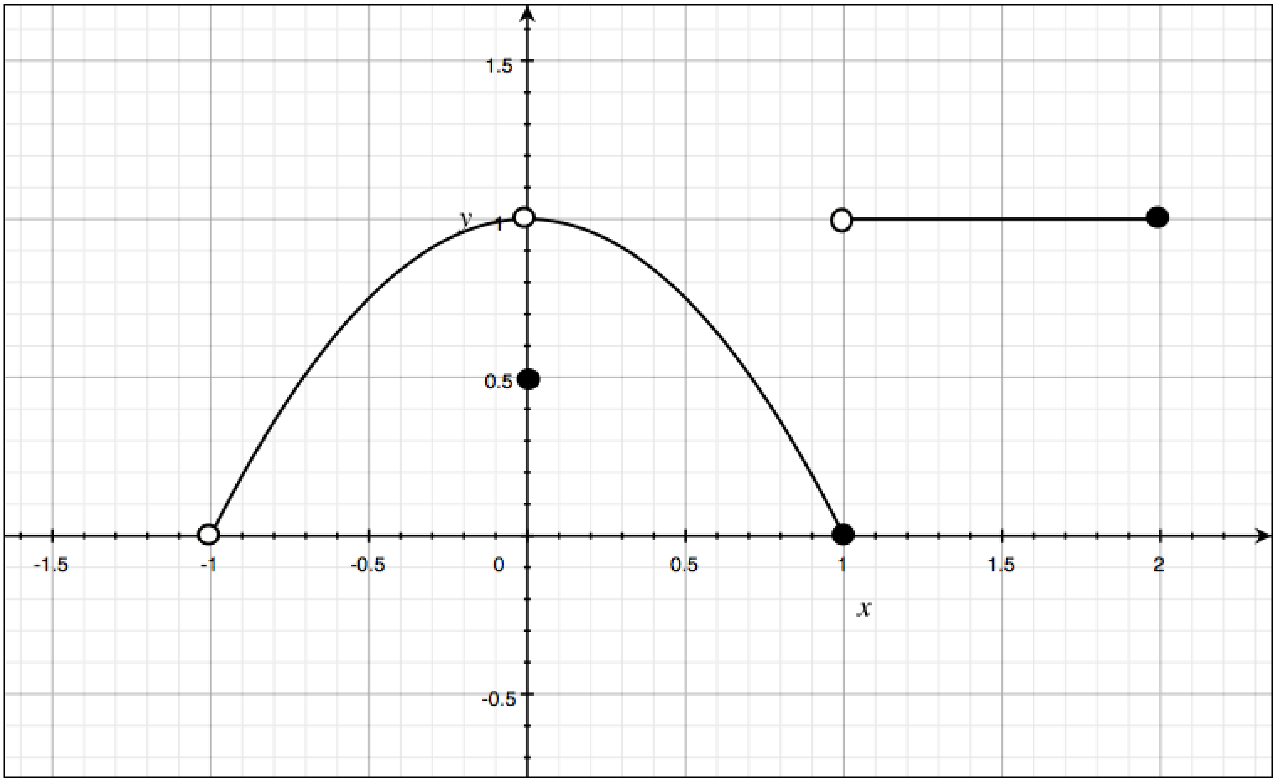
**Determine each limit.**

8. $\lim_{x \to 1^-} \text{int}(x)$

9. $\lim_{x \to 0^+} \frac{x}{|x|}$

10. $\lim_{x \to -0.001} \text{int}(x)$

Which of the statements are true about the function $y = f(x)$ graphed below and which are false?




11. $\lim_{x \to -1^+} f(x) = 0$

12. $\lim_{x \to 0^-} f(x) = 1$

13. $\lim_{x \to 0^+} f(x) = \frac{1}{2}$

14. $\lim_{x \to 1^+} f(x) = 1$

15. $\lim_{x \to 1^-} f(x) = 1$

16. $\lim_{x \to 0} f(x) = 1$

17. $\lim_{x \to 1} f(x) = 1$

18. $\lim_{x \to 2^-} f(x) = 1$

19. There is a removable discontinuity at $x = 0$.

20. There is a jump discontinuity at $x = 1$.

Given that $\lim_{x\to -2} f(x) = 0$ and $\lim_{x\to -2} g(x) = 4$ determine the following limits.

21. $\lim_{x\to -2}(g(x)+2)$

22. $\lim_{x\to -2} g^2(x)$ (Note: $g^2(x) = [g(x)]^2$)

23. $\lim_{x\to -2}(x\cdot f(x))$

24. $\lim_{x\to -2} \dfrac{f(x)}{g(x)}$

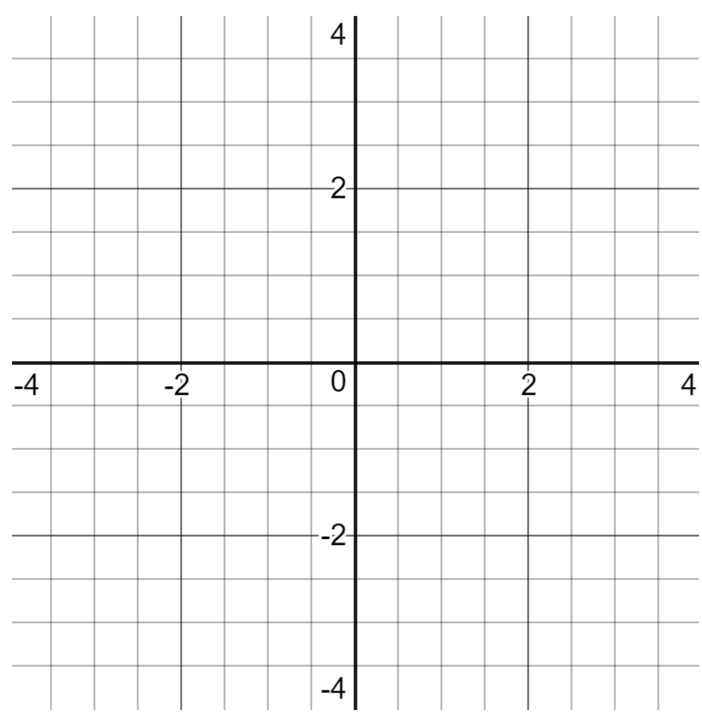
Complete the following parts for each of questions 25 and 26.

a) Draw the graph of *f*(*x*).

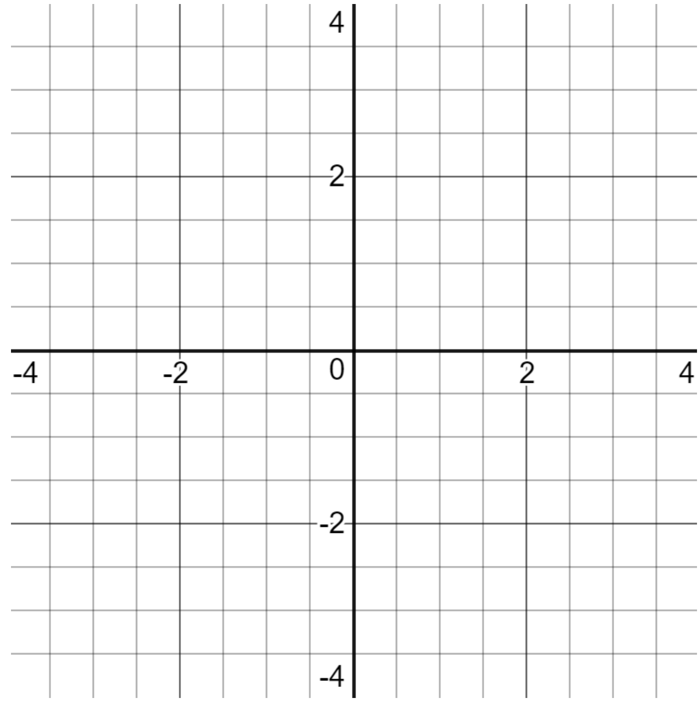
b) Determine $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.

c) Does $\lim_{x\to c} f(x)$ exist? If so, what is it? If it does not exist explain why.

25. $c = 1$, $f(x) = \begin{cases} 2+x, x<1 \\ 4-x, x>1 \end{cases}$



26. $c = 2$ , $f(x) = \begin{cases} 3 - x, & x < 2 \\ 5, & x = 1 \\ \dfrac{x}{2} - 1, & x > 2 \end{cases}$




Complete the following parts for question 27.

a) Draw the graph of *f*(*x*).

b) At what points *c* in the domain of *f* does $\lim_{x \to c} f(x)$ exist?

c) At what point "*c*" does the limit exist and why?

27. $f(x) = \begin{cases} \cos x, & 0 < x \le 2\pi \\ \sin x, & -2\pi \le x \le 0 \end{cases}$